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A Module Theory Approach on Generalization of Maschke Theorem and Schur's Lemma in Representation Theory*

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Abstract. In this work we study representations of finite abelian groups over module over a principal ideal domain. Let G be a finite abelian group and M a module over a principal ideal domain R. A representation of G over M is a group homomorphism from G to the automorphisms on M over R. We use the fact that this M can be represented as an $R[G]$-module to generalize Maschke Theorem and Schur's Lemma.

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1. Introduction

Let G be a finite abelian group and V a vector space over a field F. A representation of G over V is a group homomorphism $\rho : G \to \mathrm{Lin}_F^*(V)$, where $\mathrm{Lin}_F^*(V)$ is a group of all isomorphisms on V. For some elementary notions of representation of a group over a vector space we refer to Steinberg [10]. Some of the important results in representation theory are Maschke Theorem and Schur's Lemma, which are applied to look the properties of character. The notion of character is well known the MacWilliams Identity. Some elementary properties of linear codes and MacWilliams Identity can be refer to Ling [6] and Pellikaan et al. [9]. The studies of linear codes into more general situation have been done by some authors. Wood [11] has generalized codes as submodules of modules over Frobenius rings and has defined the dual codes.

The weight distributions and MacWilliams Identity of linear codes over finite Frobenius rings have been investigated by Byrne [3]. Then Greferath et al. [8] have continued to observe MacWilliams Identity for linear codes over Frobenius modules and quasi-Frobenius modules. One of the results of Gluesing-Luerssen [7] is for any finite Frobenius ring, the weight homogeneous denumerator can be described based on its socle and radical. Moreover, Adkins and Weintraub in [2] and James and Liebeck in [5] used module theory approach to analyse representations of groups over vector spaces. If there exists a representation of finite group G over a vector space over a field F, then V can be represented as an $F[G]$-module.

In this work we study the representation of finite abelian groups on a free module over a principal ideal domain with finite dimension. Let G be a finite abelian group and M a module over a principal ideal domain R. A representation of G over M is a group homomorphism $\rho : G \to \mathrm{Aut}_R(M)$, where $\mathrm{Aut}_R(M)$ is a group of automorphims on M. If there exists a representation of finite group G over a module over a ring R, then M can be represented as an $R[G]$-module. The main results of our work are the generalization of Maschke Theorem and Schur's Lemma.

2. Main Result

Throughout this paper the ring R is a principal ideal domain and the module M is a free left R-module with finite dimension.

Definition 2.1. *Let G be a finite abelian group and M an R-module. A representation of G over M is a group homomorphism $\rho : G \to \mathrm{Aut}_R(M)$, where $\mathrm{Aut}_R(M)$ is the set of the automorphisms on M.*

We give the example of representation of the permutation group S_3 over the $\mathbb{Z}$-module $\mathbb{Z}^3$ as following.

Example 2.2. We consider the permutation group S_3 and define $\rho\colon S_3 \to \mathrm{Aut}(\mathbb{Z}^3)$ as following: $\forall(a_1, a_2, a_3) \in \mathbb{Z}^3$,

g	(1)	(1 2)	(1 3)	(2 3)	(1 2 3)	(3 2 1)
$\rho_g((a_1, a_2, a_3))$	(a_1, a_2, a_3)	(a_2, a_1, a_3)	(a_3, a_2, a_1)	(a_1, a_3, a_2)	(a_3, a_1, a_2)	(a_2, a_3, a_1)

The proof that ρ is a group homomorphism is trivial.

Let G be a finite group. We recall the following group ring denoted by $R[G] = \{\sum_{g\in G} a_g g \mid a_g \in R, g \in G\}$ by operations addition and multiplication as follow:

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G} (a_g + b_g)g;$$

$$(\sum_{g\in G} a_g g)(\sum_{h\in G} b_h h) = \sum_{k\in G}(\sum_{gh=k} a_g b_h)k.$$

Moreover, if there exists a representation of finite group G over an R-module M, then M can be represented as an $R[G]$-module. Let G be a finite group. If ρ is a representation of G over an R-module M, where R is a commutative ring with unit, then M is an $R[G]$-module by the following scalar multiplication:

$$(\sum_{g\in G} a_g g)m = \sum_{g\in G} a_g \rho_g(m),$$

for all $\sum_{g\in G} a_g g \in R[G]$ and $m \in M$. Conversely, if M is an $R[G]$-module and $\mathcal{B}$ is a basis of M, then the function $\mu : g \mapsto [g]_{\mathcal{B}}$, where $[g]_{\mathcal{B}}$ is the matrix of the endomorphism of $m \mapsto mg$, for all $g \in G$, is a representation of G in M. For a detail explanation of this notion, we refer to Theorem 4.4 of [5].

Based on the discussion above, we conclude that a mapping $T\colon M \to N$ is an $R[G]$-homomorphism if and only if

(1) T is an R-homomorphism;
(2) $T(g.m) = g.(T(m))$ for all $g \in G$.

Definition 2.3. *Let G be a finite abelian group, M an R-module and $\rho : G \to \mathrm{Aut}_R(M)$ a representation of G. An R-submodule K of M is called a G-invariant submodule if $\rho_g(K) \subseteq K$ for any $g \in G$.*

From Definition 2.3 it is understood that every G-invariant submodule K of M is an $R[G]$-submodule of M.

A nonzero R-module M is called a simple or irreducible module if it has only the trivial submodules. An R-module M is decomposable if $M = N_1 \oplus N_2$, where N_1, N_2 are nontrivial submodules of M. An R-module M is called completely reducible if M can be represented as finite direct sum of simple submodules.

Definition 2.4. *Let $\rho\colon G \to \mathrm{Aut}_R(M)$ be a representation of finite group G in a module M over R.*

(1) *ρ is said to be irreducible if M is irreducible as an $R[G]$-module.*
(2) *ρ is said to be decomposable if $M = N_1 \oplus N_2$ where N_1, N_2 are G-invarian submodules. Moreover, if ρ is decomposable, then M is decomposable to $R[G]$-submodules N_1 and N_2.*
(3) *ρ is said to be completely reducible if $M = N_1 \oplus N_2 \oplus ... \oplus N_k$, where N_i is a G-invariant submodule which is irreducible for all $i = 1, 2, ..., k$. Moreover, if ρ is completely reducible, then $M = N_1 \oplus N_2 \oplus ... \oplus N_k$, where N_i is an $R[G]$-submodule which is irreducible for all $i = 1, 2, ..., k$.*

We give now a sufficient condition for an $R[G]$-submodule of an $R[G]$-module to be a direct summand.

Proposition 2.5. *Let R be a principal ideal domain, G a finite abelian commutative group and M be an $R[G]-$ module. If N is a nontrivial $R[G]-$ submodule of M with $|G|$ is invertible in R, then there is a nontrivial $R[G]-$ submodule K of M such that*

$$M = N \oplus K.$$

Proof. Suppose that N is an $R[G]-$ submodule of an $R[G]-$ module M. Because M is free module over PID R, so there is a submodule K_0 of M, such that $M = N \oplus K_0$. For any $m \in M$, we have $m = n + k$ for a unique $n \in N$ and $k \in K_0$. We define $\psi : M \to M$ with $\psi(m) = n$ for any $m \in M$. It is clear that Im $\psi = N$ and Ker $\psi = K_0$. We want to modify the projection ψ to get an $R[G]-$ homomorphism which image is N. Define $\phi : M \to M$ as for any $m \in M$:

$$\phi(m) = \frac{1}{|G|} \sum_{g \in G} g\psi g^{-1}(m) = \frac{1}{|G|} \sum_{g \in G} g\psi(g^{-1}m)$$

Because $|G|$ is invertible in R, $g^{-1}m \in M$ and ψ is projection, so $\psi(g^{-1}m) \in N$. It means $Im\ \phi \subseteq N$.

Claim 1. Function ϕ is an $R[G]-$ module homomorphism.

It is easy to prove $\phi(m + n) = \phi(m) + \phi(n)$ for any $m, n \in M$. For $x \in G$, we have $\psi(xm) = x\psi(m)$. As g is run over all element in G, so does $h =$

$x^{-1}g, (h^{-1} = g^{-1}x)$,

$$\begin{aligned}\phi(xm) &= \frac{1}{|G|}\sum_{g\in G} g\psi g^{-1}(xm) \\ &= \frac{1}{|G|}\sum_{g\in G} g\psi(g^{-1}xm) = \frac{1}{|G|}\sum_{h\in G} xh\psi(h^{-1}m) \\ &= x\frac{1}{|G|}\sum_{h\in G} h\psi h^{-1}(m) = x\phi(m).\end{aligned}$$

Claim 2. $\phi^2 = \phi$.

For any $g \in G$ and $n \in N$, we have $gn \in N$ and $\psi(gn) = gn$, and so

$$\begin{aligned}\phi(n) &= \frac{1}{|G|}\sum_{g\in G} g\psi g^{-1}(n) = \frac{1}{|G|}\sum_{g\in G}(g\psi(g^{-1}(n)) \\ &= \frac{1}{|G|}\sum_{g\in G}(gg^{-1}n) = \frac{1}{|G|}\sum_{g\in G} n = \frac{1}{|G|}|G|n = n.\end{aligned}$$

It means $\mathrm{Im}\ \phi \supseteq N$ and moreover $\mathrm{Im}\ \phi = N$. For any $m \in M$, $\phi(m) \in N$, so $\phi(\phi(m)) = \phi(m)$. Let $K = \mathrm{Ker}\ \phi$. Then K is $R[G]-$ submodule and because ϕ projection $M = N \oplus K$. ∎

Proposition 2.6. (Generalized Maschke Theorem) *Let R be a principal ideal domain, M a free R-module with finite dimension and G an abelian commutative group. Let $\rho\colon G \to \mathrm{Aut}_R(M)$ be a representation of G over M. If $|G|$ is invertible in R, then ρ is completely reducible.*

Proof. We use the mathematical induction to the dimension of the module to prove this proposition. If $\dim(M) = 1$, then ρ is irreducible. Now we assume that the assertion is true for $\dim(M) \leq n$. We show that if $\dim(M) = n + 1$, then ρ is completely reducible.

If ρ is irreducible, then we prove the proposition. If ρ is reducible, then base on Proposition 2.5 $M = N_1 \oplus N_2$ where N_1, N_2 are $R[G]$-submodules of M. Since R is a principal ideal domain, $\dim(N_1), \dim(N_2) < \dim(M) = n + 1$. By assumption $N_1 = U_1 \oplus U_2 \oplus ... \oplus U_r$ and $N_2 = V_1 \oplus V_2 \oplus ... \oplus V_s$, where U_i, V_j are irreducible for all $i = 1, 2, ..., r$ and $j = 1, 2, ..., s$. Hence we have

$$M = U_1 \oplus U_2 \oplus ... \oplus U_r \oplus V_1 \oplus V_2 \oplus ... \oplus V_s.$$

So ρ is completely reducible. ∎

Let G be a finite abelian group, R a principal ideal domain and M, N R-modules. We give $\rho\colon G \to \mathrm{Aut}_R(M)$ and $\varphi\colon G \to \mathrm{Aut}_R(N)$ two representations of G over M. We denote $\rho_g\colon M \to M$ and $\varphi_g\colon N \to N$ as module isomorphisms for any $g \in G$.

For any module homomorphism $T\colon M \to N$, $T\rho_g\colon M \to N$ and $\varphi_g T\colon M \to N$ are module homomorphisms, but it is not necessary $T\rho_g = \varphi_g T$. We give the following example to show this fact.

Example 2.7. We consider Example 2.2. Moreover, we also consider the following homomorphism $\varphi\colon S_3 \to \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^2)$ which is defined as $\varphi_g = id$, the identity on $\mathbb{Z}^2$, for all $g \in S_3$. It is easy to prove that ρ and φ are representations of S_3 with degree 3 and 2. Let $T\colon \mathbb{Z}^3 \to \mathbb{Z}^2$ be a $\mathbb{Z}$-module homomorphism with definition

$$T(a_1, a_2, a_3) = (a_1, a_2 + a_3).$$

Since for $(1\ 2) \in S_3$ we have

$$\begin{aligned} T\rho_{(1\ 2)}(a_1, a_2, a_3) &= (a_2, a_1 + a_3) \\ \varphi_{(1\ 2)}T(a_1, a_2, a_3) &= (a_1, a_2 + a_3) \end{aligned}$$

for $\forall (a_1, a_2, a_3) \in \mathbb{Z}^3$, we conclude $T\rho_{(1\ 2)} \neq \varphi_{(1\ 2)}T$. Thus there exists $g \in S_3$ such that $T\rho_g \neq \varphi_g T$.

Furthermore, we give the following definition.

Definition 2.8. *Let G be a finite abelian group, R a principal ideal domain, M, N R-modules. Let $\rho\colon G \to \mathrm{Aut}_R(M)$ and $\varphi\colon G \to \mathrm{Aut}_R(N)$ be two representations of G with finite degree. A module homomorphism $T\colon M \to N$ is called a morphism from ρ to φ, if T satisfies $T\rho_g = \varphi_g T$ for all $g \in G$.*

Now we give the example of morphism between two representations.

Example 2.9. Recall two representations ρ and φ in Example 2.7. Now we define an invertible $\mathbb{Z}$-module homomorphism $T\colon \mathbb{Z}^3 \to \mathbb{Z}^2$ where $T(a_1, a_2, a_3) = (-a_3, a_2, -a_1)$, for all $(a_1, a_2, a_3) \in \mathbb{Z}^3$. We also define $\varphi_g = T\rho_g T^{-1}$ for all $g \in S_3$, and we obtain

g	(1)	$(1\ 2)$	$(1\ 3)$
$\rho_g((a_1, a_2, a_3))$	(a_1, a_2, a_3)	$(a_1, -a_3, -a_2)$	(a_3, a_2, a_1)

g	$(2\ 3)$	$(1\ 2\ 3)$	$(3\ 2\ 1)$
$\rho_g((a_1, a_2, a_3))$	$(-a_2, -a_1, a_3)$	$(-a_2, -a_3, a_1)$	$(a_3, -a_1, -a_2)$

It can be proved that $T\rho_g = \varphi_g T$ for all $g \in S_3$. Hence T is a morphism from ρ to φ.

We consider again the notions in Definition 2.8. The set of all morphisms from ρ to φ is denoted by $\mathrm{Hom}_G(M, N)$ and it is understood that $\mathrm{Hom}_G(\rho, \varphi) \subseteq \mathrm{Hom}_R(M, N)$. Since R is a principal ideal domain, $\mathrm{Hom}_R(M, N)$ is an R-module.

Proposition 2.10. *Let M, N be R-modules, $\rho : G \to \mathrm{Aut}_R(M)$ and $\varphi : G \to \mathrm{Aut}_R(N)$ be representations of finite abelian group G. Then $\mathrm{Hom}_G(\rho, \varphi)$ is a submodule of $\mathrm{Hom}_R(M, N)$.*

Proof. The zero homomorphism is always contained in $\mathrm{Hom}_G(\rho, \varphi)$, so it is a non-empty set. Take any T_1, T_2 in $\mathrm{Hom}_G(\rho, \varphi)$, r, s in R and g in G. Then we have

$$(rT_1 + sT_2)\rho_g = rT_1\rho_g + sT_2\rho_g = r\varphi_g T_1 + s\varphi_g T_2 = \varphi_g(rT_1 + sT_2).$$

Hence $\mathrm{Hom}_G(\rho, \varphi)$ is a submodule of $\mathrm{Hom}_R(M, N)$. ∎

Definition 2.11. *Two representations of G, $\rho : G \to \mathrm{Aut}_R(M)$ and $\varphi : G \to \mathrm{Aut}_R(N)$, are said to be equivalent, denoted by $\rho \sim \varphi$, if there exists an R-isomorphism $T : M \to N$ such that $T\rho_g = \varphi_g T$ for all $g \in G$.*

It is clear from Definition 2.11 that the following proposition holds.

Proposition 2.12. *Let $\rho : G \to \mathrm{Aut}_R(M)$ and $\varphi : G \to \mathrm{Aut}_R(N)$ be representations of finite abelian group G. Then $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$ are G-invariant submodules.*

Proof. Take any $x \in \mathrm{Ker}(T)$ and $g \in G$. Then $(T\rho_g)(x) = (\varphi T)(x) = 0$ and we conclude that $\rho_g)(x) \in \mathrm{Ker}(T)$. Now take any $y \in \mathrm{Im}(T)$. Then there is $m \in M$ such that $T(m) = y$. Moreover, $\rho_g(y) = (\rho_g T)(m) = (T\varphi_g)(m) \in \mathrm{Im}(T)$. ∎

Now we give a generalized Schur's Lemma as following.

Proposition 2.13. (Generalized Schur's Lemma) *Let $\rho : G \to \mathrm{Aut}_R(M)$ and $\varphi : G \to \mathrm{Aut}_R(N)$ be representations of finite abelian group G and $T \in \mathrm{Hom}_G(\rho, \varphi)$. Then $T = 0$ or T is invertible. Moreover,*

(i) *if $\rho \not\sim \varphi$, then $\mathrm{Hom}_G(\rho, \varphi) = 0$.*

(ii) *if $\rho = \varphi$, and K is the fractional field of R, K' is the algebraic extension field of K, then $T = \lambda I$ with $\lambda \in K'$.*

Proof. If $T = 0$, then it is clear. Now we assume $T \neq 0$. Based on Proposition 2.12, $\mathrm{Ker}(T)$ is G-invariant, so either $\mathrm{Ker}(T) = M$ or $\mathrm{Ker}(T) = 0$. But $T \neq 0$, hence $\mathrm{Ker}(T) = 0$, i.e. T is a monomorphism. Again we apply Proposition 2.12, $\mathrm{Im}(T)$ is G-invariant, so either $\mathrm{Im}(T) = M$ or $\mathrm{Im}(T) = 0$. But $T \neq 0$, hence $\mathrm{Im}(T) = M$, i.e. T is an epimorphism. We conclude that T is invertible.

For (i), we suppose that $\mathrm{Hom}_G(\rho, \varphi) \neq 0$, say there is a nonzero homomorphism $T \in \mathrm{Hom}_G(\rho, \varphi)$. But it will imply T is invertible, so we obtain that $\rho \sim \varphi$, a contradiction.

For (ii), let $\lambda \in K'$ be eigenvalue of T. Consider that R is a principle ideal domain. Then by Proposition 16.3.14 and Definition of Dedekind Domain in

[4] R is integrally closed in K. Thus if $\lambda \in K$ then $\lambda \in R$. By definition of an eigenvalue $\lambda I - T$ is not invertible. Consider that $I \in Hom_G(\rho, \rho)$. By Proposition 2.10, I is a submodule of $Hom_R(M, M)$) and $\lambda I - T$ belongs to $Hom_G(\rho, \rho)$. Since all non zero element in $Hom_G(\rho, \rho)$ is invertible by the first paragraph of the proof, hence $\lambda I - T = 0 \Leftrightarrow T = \lambda I$. Furthermore if $\lambda \notin K$ then $T = \lambda I$ with $\lambda \in K'$. ∎

Remark 2.14. If $\rho : G \to \mathrm{Aut}_R(M)$ and $\varphi : G \to \mathrm{Aut}_R(N)$ are equivalent irreducible representations of finite abelian group G, then

$$\dim(Hom_G(\rho, \varphi)) = 1.$$

Now we describe the irreducible representation of an abelian group.

Corollary 2.15. *Let G be an abelian group. Then any irreducible representation of G has degree one.*

Proof. Let $\rho : G \to Aut_R(M)$ be an irreducible representation, $h \in G$ and setting $T = \rho_h$. We obtain for all $g \in G$ that

$$T\rho_g = \rho_h\rho_g = \rho_{hg} = \rho_{gh} = \rho_g\rho_h.$$

So we have $\rho_h \in Hom_G(\rho, \rho)$. Consequently, using Proposition 2.13 we have $\rho_h = \alpha_h I$ for some scalar $\alpha_h \in K$ (this α is dependence on h). Since R is integrally closed in K, we conclude $\alpha \in R$. Let m be a non-zero element in M and $k \in R$. Then $\rho_h(km) = \alpha_h Ikm = \alpha_h km \in Rm$. Thus Rm is a G-invariant submodule, as h was arbitrary. We conclude that $M = Rm$ by irreducibility and $\dim(M) = 1$. ∎

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